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Stochastic calculus in superspace: II. Differential forms, supermanifolds and the Atiyah–Singer index theorem

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Abstract. Starting with vector bundles over manifolds, supermanifolds are constructed whose function algebras correspond to twisted differential forms. Stochastic calculus for bosonic and fermionic Brownian paths is used to provide a geometric construction of Brownian paths on these supermanifolds. A Feynman–Kac formula for the heat kernel of the Laplace–Beltrami operator is then derived. This is used to provide a simple, rigorous version of the supersymmetric proofs of the Atiyah–Singer index theorem.

1. Introduction

In this paper superspace stochastic calculus is used to define covariant Brownian paths on supermanifolds, and thus to extend fermionic path integration to curved space. The construction is used to study the Laplace–Beltrami operator on differential forms on a Riemannian manifold.

The stochastic techniques employed in this paper are somewhat different from those in other work where probabilistic methods are used to study the Laplace–Beltrami operator on forms and related objects. The key difference is the use of fermionic Brownian paths which are paths in a space of anticommuting variables. This approach, which is described in [1] and the companion paper [2], is designed to resemble as closely as possible both standard Wiener paths and the fermionic paths of the physics literature which were introduced by Martin [3] and have proved a powerful tool in heuristic calculations. Rigorous fermionic path integration along these lines has also been considered by Haba [4]. Superspace stochastic calculus considers Brownian paths in superspace, a space parametrized by both commuting and anticommuting variables. Such spaces do not directly model physical space, but are useful mathematical constructs because the spaces of functions naturally defined on them carry representations of fermionic operators defined in physics, and also (in a geometric context) of differential operators on forms on manifolds and on cross sections of spin bundles. One aim of the current paper is to show that these superspace paths, when handled in a rigorous manner, lead to new and useful analytic techniques.

In the context of conventional probability theory (without anticommuting variables) an extended study of probabilistic techniques applied to many aspects of analysis on manifolds has been made by Elworthy [5]; this work includes a straightforward probabilistic proof of the Gauss–Bonnet–Chern theorem. There are a number of other works on applications of probabilistic methods to index theory and localization, such as the work of Bismut [6], Jones and Leandre [7], Leandre [8], Lott [9] and Watanabe [10]. These other works use more technical probabilistic methods, such as Malliavin calculus, than those of this paper, where analytic estimates based on comparisons of solutions of stochastic differential equations are used. Also, in this paper heat kernels are obtained by Duhamel's formula, rather than by the use of Brownian bridges. Supermanifolds are also used in Getzler's proof of the index theorem, which uses pseudo-differential operator methods to obtain the necessary heat kernel asymptotics [11]. A detailed account of these methods may be found in the recent book of Berline *et al* [12].

When considering fermions in curved space, and differential forms on Riemannian manifolds, flat global superspace must be replaced by what is known as a supermanifold; essentially supermanifolds are extensions of ordinary manifolds to include anticommuting coordinates. Section 2 defines some supermanifolds which can be constructed in a natural way from a vector bundle over a Riemannian manifold. The construction does not depend in any essential way on which of the various approaches to supermanifolds existing in the literature is used. The supermanifolds constructed allow superspace stochastic techniques to be applied to various physical and geometric problems.

Section 3 of this paper contains a geometric formulation of Brownian paths on supermanifolds. As in the classical treatment of Brownian paths on a manifold [13, 14], this is done by constructing stochastic differential equations globally on the supermanifold. Next, using key technical results from the companion paper [2], these paths are used to give a Feynman–Kac formula for the Laplace–Beltrami operator for twisted differential forms.

In the final section of the paper this formula is used to give a rigorous version of the very simple proofs of the index theorem using supersymmetry due to Alvarez-Gaumé [15] and to Friedan and Windey [16]. In [15] and [16] the path integral calculations are carried out by physicists' methods which are not entirely rigorous, particularly in curved space; the stochastic machinery developed in this paper allows these steps to be made rigorous without spoiling the underlying simplicity and elegance of the approach.

The proofs of the index theorem given in [15] and [16] used formulae for the index of a differential operator in terms of an evolution operator $\exp -Ht$, where H is the Hamiltonian of a supersymmetric system. The first example of such a formula was given by McKean and Singer [17]; subsequently Witten showed that properties of supersymmetric quantum mechanics could be used to derive many analogous formulae [18]. The evolution operator was then expressed in terms of path integrals. Using techniques which are not fully rigorous, it was then shown that, for the purposes of calculating the index, the complicated curved-space Hamiltonian H could be replaced by a much simpler flat-space supersymmetric magnetic oscillator Hamiltonian whose evolution operator was then calculated exactly by standard methods. It is the validity of this approximation which is established by the stochastic calculus techniques developed in this paper. The approach taken in this paper thus retains much of the simplicity of the original supersymmetric proofs [15, 16].

2. Supermanifolds and differential forms

This section is purely geometrical, constructing supermanifolds which provide the appropriate arena for the Brownian paths of sections 3 and 4. The supermanifolds are constructed using the data of a vector bundle over a conventional manifold in such a way that the space of supersmooth functions on this supermanifold is isomorphic to the space of twisted differential forms on the manifold. In section 3 this construction is used to transfer superspace Brownian motion to bundles of twisted differential forms.

A supermanifold is a space which has some commuting and some anticommuting coordinates. There are a number of different approaches to supermanifolds in the literature, which are broadly equivalent; the constructions in this paper do not depend on the detailed aspects of any particular approach. The important fact is that all local coordinates belong to a graded commutative algebra, with even elements (which commute with elements of either parity) represented by lower-case Roman letters and odd elements (which commute with even elements but anticommute with any odd element) denoted by Greek letters. The dimension of a supermanifold is an ordered pair of integers which specifies the number of even and odd local coordinates. Some facts about the analysis of commuting and anticommuting variables are given in section 2 of the companion paper [2]. More detailed accounts may be found in [19]. As in the case of conventional manifolds, a supermanifold can be entirely characterized by its coordinate transition functions, an approach that is used in this section. Details of the reconstruction of a supermanifold from its transition functions may be found in [20].

Let M be a smooth, compact m-dimensional real manifold and let E be a smooth n-dimensional Hermitian vector bundle over M. Suppose that $\{U_\alpha | \alpha \in \Lambda\}$ is an open cover of M by sets which are both coordinate neighbourhoods of M and local trivialization neighbourhoods of E. For each $\alpha \in \Lambda$ let $\phi_\alpha : U_\alpha \to \mathbb{R}^m$ be the coordinate map on U_α, and for each $\alpha, \beta \in \Lambda$ let $h_{\alpha\beta} : U_\alpha \cap U_\beta \to U(n)$ be the transition function of the bundle E (so that for each $p \in U_\alpha \cap U_\beta$ $\left(h_{\alpha\beta}{}^r{}_s(p)\right)$ is a unitary $n \times n$ matrix). Additionally let $\{\tau_{\alpha\beta} | \alpha, \beta \in \Lambda\}$ be the coordinate transition functions on M, that is $\tau_{\alpha\beta} : \phi_\beta(U_\alpha \cap U_\beta) \to \phi_\alpha(U_\alpha \cap U_\beta)$ with $\tau_{\alpha\beta} = \phi_\alpha \circ \phi_\beta^{-1}$, and let $\left(m_{\alpha\beta}{}^i{}_k(x_\beta)\right) = \left(\partial x^i_\alpha / \partial x^k_\beta\right)$ be the corresponding Jacobian matrix.

The required supermanifold $S(E)$ is the $(m, m+n)$-dimensional supermanifold built over M with local coordinates $(x^1_\alpha, \ldots, x^m_\alpha, \theta^1_\alpha, \ldots, \theta^m_\alpha, \eta^1_\alpha, \ldots, \eta^n_\alpha)$ and transition functions

$$\begin{aligned} T_{\alpha\beta} : (x^1_\beta, \ldots, x^m_\beta, \theta^1_\beta, \ldots, \theta^m_\beta, \eta^1_\beta, \ldots, \eta^n_\beta) \\ \mapsto \left(\tau^1_{\alpha\beta}(x_\beta), \ldots, \tau^m_{\alpha\beta}(x_\beta), m_{\alpha\beta}{}^1{}_k(x_\beta)\theta^k_\beta, \ldots, m_{\alpha\beta}{}^m{}_k(x_\beta)\theta^k_\beta,\right. \\ \left. h_{\alpha\beta}{}^1{}_r(\phi_\beta^{-1}(x_\beta))\eta^r_\beta, \ldots, h_{\alpha\beta}{}^n{}_r(\phi_\beta^{-1}(x_\beta))\eta^r_\beta\right). \end{aligned} \tag{2.1}$$

(Here and elsewhere the convention that repeated indices are to be summed over their range is used.)

A useful space of functions is the space $C^{\infty\prime}(S(E))$ of functions f which locally take the form

$$f(x, \theta, \eta) = \sum_{\mu \in M_m} \sum_{r=1}^{n} f_{\mu r}(x)\theta^\mu \eta^r \tag{2.2}$$

where $\mu = \mu_1 \cdots \mu_k$ is a multi-index with $1 \leqslant \mu_1 < \cdots < \mu_k \leqslant m$, M_m is the set of all such multi-indices (including the empty one), $\theta^\mu = \theta^{\mu_1} \cdots \theta^{\mu_k}$ and each $f_{\mu r}$ is in

$C^\infty(\mathbb{R}^m, \mathbb{C})$. (It should be noted that these functions are linear in the η^r but may be multilinear in the θ^i; the nature of the transition functions (2.1) ensure that such functions may be consistently defined.) Now it may be seen that, again as a result of the choice of transition functions (2.1), there is a globally defined map

$$I : \Gamma(\Omega(M) \otimes E) \to C^{\infty\prime}(S(E), \mathbb{C})$$

which may be obtained from the local prescription

$$I(s)(x, \theta, \eta) = \sum_{\mu \in M_m} \sum_{r=1}^{n} f_{\mu r}(x) \theta^\mu \eta^r \qquad \text{if} \qquad s(x) = \sum_{\mu \in M_m} \sum_{r=1}^{n} f_{\mu r}(x) \mathrm{d}x^\mu e^r \tag{2.3}$$

where $\Omega(M)$ is the bundle of smooth forms on M and $(e^1, \ldots, e^n)$ is the appropriate basis of the fibre of the bundle E. Also, this map is an isomorphism of vector spaces, and indeed of sheaves.

The crucial feature of this construction, which is itself quite simple, is that it allows one to express the Hodge–de Rham operator on twisted differential forms on M as a differential operator on the extended space of functions $C^{\infty\prime}(S(E), \mathbb{C})$. Fermionic path integration techniques then allow one to use stochastic methods to analyse these operators in the usual way.

Explicitly, suppose that M is a Riemannian manifold with metric g, and that a connection has been chosen on the bundle E. Then, in local coordinates, if one introduces the notation $\partial_i = \partial/\partial x^i$, $\delta_{\theta^i} = \partial/\partial\theta^i$ and $\delta_{\eta^r} = \partial/\partial\eta^r$, the Hodge–de Rham operator $d + \delta$ takes the form

$$d + \delta = (\theta^i - g^{ij}\delta_{\theta^j})\partial_i - g^{il}\Gamma_{ij}{}^k \theta^j \delta_{\theta^l}\delta_{\theta^k} + (\theta^i - g^{ij}\delta_{\theta^j}) A_i{}_r{}^s \eta^r \delta_{\eta^s} . \tag{2.4}$$

where $\Gamma_{ij}{}^k$ are the Christoffel symbols of the Riemannian connection on (M, g) and $A_i{}_r{}^s$ are the components of the connection one form on E pulled back to the coordinate neighbourhood by the section corresponding to the local trivialization. Using the notation $\psi^i = \theta^i - g^{ij}\partial/\partial\theta^j$, this takes the simpler form

$$d + \delta = \psi^i(\partial_i - \Gamma_{ij}{}^k \theta^j \delta_{\theta^k} - A_i{}_r{}^s \eta^r \delta_{\eta^s}) . \tag{2.5}$$

As will emerge in section 5, it is the heat kernel of the square of this operator, the Laplace–Beltrami operator, which can be studied by fermionic stochastic calculus, and is relevant to the proof of the Atiyah–Singer index theorem. For these purpose it is useful to establish the following lemma, which is a twisted version of the Weitzenbock formula relating the Bochner Laplacian to the Laplace–Beltrami operator. The proof uses the method of Cycon *et al* [21], generalized to the twisted case. (A normalization factor of $\frac{1}{2}$ is included in the Laplace–Beltrami operator to conform with the standard normalization of Brownian motion.)

Lemma 2.1. Let $L = \frac{1}{2}(d + \delta)^2$ be the Laplace–Beltrami operator on the space of functions $C^{\infty\prime}(S(E), \mathbb{C})$. Then

$$L = -\tfrac{1}{2}(B - R_i^j(x)\theta^i \delta_{\theta^j} - \tfrac{1}{2} R_{ki}{}^{jl}(x) \theta^i \theta^k \delta_{\theta^j} \delta_{\theta^l} + \tfrac{1}{4}[\psi^i, \psi^j] F_{ij}{}_r{}^s(x) \eta^r \delta_{\eta^s}) \tag{2.6}$$

where $R_{ki}{}^{mq}$ are the components of the curvature of (M,g), $F_{ijr}{}^{s}$ are the components of the curvature of the connection on E, and B is the twisted Bochner Laplacian,

$$B = g^{ij}(D_i D_j - \Gamma_{ij}{}^{k} D_k) \tag{2.7}$$

with

$$D_i = \partial_i - \Gamma_{ij}{}^{k}\theta^j \delta_{\theta^k} - A_{i\ r}{}^{s}\eta^r \delta_{\eta^s}. \tag{2.8}$$

Proof.

$$\begin{aligned} L &= \tfrac{1}{2}(\psi^i D_i \psi^j D_j) \\ &= \tfrac{1}{2}\big((\tfrac{1}{2}\{\psi^i,\psi^j\} + \tfrac{1}{2}[\psi^i,\psi^j])D_i D_j + \psi^i[D_i,\psi^j]D_j\big) \end{aligned} \tag{2.9}$$

(where $\{,\}$ denotes an anticommutator and $[,]$ denotes a commutator). Now

$$\tfrac{1}{2}\{\psi^i,\psi^j\} = -g^{ij} \tag{2.10}$$

while

$$\begin{aligned} &\tfrac{1}{2}[\psi^i,\psi^j]D_i D_j \\ &\quad = -\tfrac{1}{2}[\psi^i,\psi^j]\big((\partial_i \Gamma_{jk}{}^{\ell} - \Gamma_{ik}{}^{\ell'}\Gamma_{j\ell'}{}^{m})\theta^k \delta_{\theta^\ell} \\ &\qquad + (\partial_i A_{j\ r}{}^{s} - A_{i\ r}{}^{t} A_{j\ t}{}^{s})\eta^r \delta_{\eta^s}\big) \\ &\quad = R_i^j(x)\theta^i \delta_{\theta^j} + \tfrac{1}{2}R_{ki}{}^{j\ell}(x)\theta^i\theta^k \delta_{\theta^j}\delta_{\theta^\ell} - \tfrac{1}{4}[\psi^i,\psi^j]F_{ijr}{}^{s}(x)\eta^r \delta_{\eta^s}. \end{aligned} \tag{2.11}$$

Also

$$\begin{aligned} [D_i,\psi^j] &= -\partial_i g^{jk}\delta_{\theta^k} - \Gamma_{ik}{}^{j}\theta^k - \Gamma_{ik}{}^{\ell} g^{jk}\delta_{\theta^\ell} \\ &= -\Gamma_{ik}{}^{j}\psi^k. \end{aligned} \tag{2.12}$$

Thus

$$\begin{aligned} \psi^i[D_i,\psi^j]D_j &= -\psi^i\psi^k\Gamma_{ik}{}^{j}D_j \\ &= g^{ik}\Gamma_{ik}{}^{j}D_j. \end{aligned} \tag{2.13}$$

Hence

$$\begin{aligned} L = -\tfrac{1}{2}\big(&g^{ij}D_iD_j - g^{ij}\Gamma_{ij}{}^{k}D_k - R_i^j(x)\theta^i\delta_{\theta^j} - \tfrac{1}{2}R_{ki}{}^{j\ell}(x)\theta^i\theta^k\delta_{\theta^j}\delta_{\theta^\ell} \\ &+ \tfrac{1}{4}[\psi^i,\psi^j]F_{ijr}{}^{s}(x)\eta^r\delta_{\eta^s}\big) \end{aligned} \tag{2.14}$$

as required.

One further supermanifold is required for the stochastic constructions in the following section. This is a super extension $S(O(M), E)$ of the bundle of orthonormal frames on the m-dimensional manifold M (again twisted according to the Hermitian bundle E). The supermanifold $S(O(M), E)$ can be defined by its coordinate transition functions; suppose that $\{U_\alpha | \alpha \in \Lambda\}$ is again an open cover of M by sets which are both coordinate neighbourhoods of M and local trivialization neighbourhoods of E. Let $x^i_\alpha, b^u_\alpha, i = 1, \ldots, m, u = 1, \ldots, \frac{1}{2}m(m-1)$ be local coordinates on $O(M)|_{U_\alpha}$. Then $S(O(M), E)$ is the $[\frac{1}{2}m(m+1), m+n]$-dimensional supermanifold with local coordinates $x^i_\alpha, b^u_\alpha, \theta^j_\alpha, \eta^r_\alpha$. On overlapping neighbourhoods U_α and U_β the coordinates x^i, θ^j, η^r have the transition functions defined by (2.1), while the coordinates b^u transform as on the bundle of orthonormal frames $O(M)$.

It should be noted that anti-commuting variables are introduced in order to provide function algebras which have geometric and physical applications; anticommuting variables do not directly model physical situations. An important algebraic tool is the Berezin integral [22] which integrates out the anticommuting variables to give a real or complex number. This is defined in the following manner. Suppose that f is a polynomial function of p anticommuting variables $\alpha^1, \ldots, \alpha^p$. Then

$$f(\alpha) = k\alpha^1\alpha^2 \ldots \alpha^p + \text{lower-order terms} \tag{2.15}$$

where k is a real or complex number. The Berezin integral is then defined by

$$\int_B f(\alpha) = k\,. \tag{2.16}$$

The integral defined by this simple prescription has a number of useful properties which will be exploited in later sections of this paper.

3. Geometric Brownian paths on supermanifolds

This section constructs Brownian paths on the supermanifolds introduced in section 2. Before discussing these generalized Brownian paths in curved superspace, a brief review of path integral techniques for classical manifolds is given.

Clearly it is not straightforward to transfer Brownian motion to the setting of a general manifold, because Brownian motion, while invariant under rigid Euclidean transformations, will not survive more general coordinate transformations. Various analogues of Brownian motion on Riemannian manifolds have been considered, two of which will be described here because they are suitable for generalization to superspace. The first method is straightforward in principle—one simply replaces the Wiener measure with a measure whose finite-dimensional marginal distributions are based on the heat kernel of the Laplacian of the manifold, an approach which leads almost trivially to a Feynman–Kac formula for a Hamiltonian which is the sum of the Laplacian and arbitrary first-order and scalar terms. This approach has been used in conjunction with fermionic Brownian motion in [23] to analyse the Hodge–de Rham operator on a Riemann manifold, and hence prove the Gauss–Bonnet–Chern formula; however its usefulness is limited by its dependence on information about the heat kernel of the Laplacian, which is itself a highly non-trivial object.

A second approach to Brownian paths on manifolds, described in [13] and [14], is to use paths which are solutions of stochastic differential equations, these being defined in a manner which is globally valid. The second-order correction term in the Itô formula (equation (3.1) of [2]) prevents one from using vector fields and other tensorial objects in the obvious way, a difficulty which is overcome by modifying the Itô integral so that a compensating term is included in the transformation rule. The most elegant way of doing this is to introduce the symmetric product or Stratonovich integral, which is defined in the following way.

Definition 3.1. Suppose that X_s and Y_s are stochastic integrals with

$$\mathrm{d}X_s = f_{a,s}\mathrm{d}b^a_s + f_{0,s}\mathrm{d}s \qquad \mathrm{d}Y_s = g_{a,s}\mathrm{d}b^a_s + g_{0,s}\mathrm{d}s\,. \tag{3.1}$$

Then

$$Y_s \circ \mathrm{d}X_s =_{\mathrm{def}} Y_s(f_{a,s}\mathrm{d}b^a_s + f_{0,s}\mathrm{d}s) + \tfrac{1}{2} f_{a,s} g_{a,s}\mathrm{d}s\,. \tag{3.2}$$

(Here b^a_s, $a = 1, \ldots, m$ denotes m-dimensional Brownian motion, and $\mathrm{d}b^a_s$ denotes an Itô differential.) As before, a repeated index is summed over its range.

Now suppose that, for $a = 0, 1, \ldots, m$, A_a are vector fields on a p-dimensional manifold N. In a local coordinate system $(x^1, \ldots, x^p)$ where $A_a = A^i_a(x)\partial/\partial x^i$ consider the stochastic differential equation

$$\mathrm{d}x^i_s = A^i_a(x_s) \circ \mathrm{d}b^a_s + A^i_0(x_s)\mathrm{d}s\,. \tag{3.3}$$

By applying the Itô calculus one finds that (3) is form-invariant under change of coordinate $\tilde{x}_s = \tilde{x}(x_s)$, where $\tilde{x}$ is a change of coordinate function. The solution to such an equation exists globally, as can be shown by a careful patching argument [13, 14]. Such equations enable one to define a notion of stochastic flow on a manifold. They also enable one to construct functions on $N \times \mathbb{R}^+$ which satisfy the differential equation

$$\frac{\partial f}{\partial t} = \frac{1}{2} A_a A_a f \qquad \text{with} \qquad f(x, 0) = h(x) \tag{3.4}$$

where h is a smooth function on N and A_a has suitable properties, as in the following theorem.

Theorem 3.2. Suppose that x_t is a solution to (3.3) with initial condition $x_0 = x \in M$. Then $f(x,t) = \mathbb{E}(h(x_t))$ satisfies the differential equation (3.4).

Outline of Proof. Using the Itô formula, and omitting details of the patching of solutions over different coordinate neighbourhoods,

$$\mathbb{E}\big(h(x_t)\big) - h(x) = \mathbb{E}\Big(\int_0^t \partial_i h(x_s)\mathrm{d}x^i_s + \int_0^t \frac{1}{2} A^i_a(x_s) A^j_a(x_s) \partial_i \partial_j h(x_s)\mathrm{d}s\Big)\,. \tag{3.5}$$

Now

$$\mathrm{d}x^i_s = A^i_a(x_s)\mathrm{d}b^a + \tfrac{1}{2} A^j_a(x_s)\big(\partial_j A^i_a(x_s)\big)\mathrm{d}s\,. \tag{3.6}$$

Thus, using the fact that Itô increments $\mathrm{d}b_s^a$ have zero expectation and are independent of b_u when $u \leqslant s$, one finds that

$$\begin{aligned}
\mathbb{E}(h(x_t)) - h(x) \\
&= \mathbb{E}\Big(\int_0^t \frac{1}{2} A_a^j(x_s)(\partial_j A_a^i(x_s))\partial_i h(x_s) \\
&\quad + \frac{1}{2} A_a^i(x_s) A_a^j(x_s)\partial_i\partial_j h(x_s)\mathrm{d}s\Big) \\
&= \int_0^t \frac{1}{2} A_a A_a h(x_s)\mathrm{d}s\,. \qquad (3.7)
\end{aligned}$$

Hence, for suitable A_a,

$$\mathbb{E}(h(x_t)) = \mathrm{e}^{\frac{1}{2}(A_a A_a)t} h(x) \qquad (3.8)$$

and thus $f(x,t) = \mathbb{E}(h(x_t))$ solves (3.3). (The x dependence of $f(x,t)$ comes from the initial condition satisfied by x_t.) □

In the case of a Riemannian manifold (M,g), when one wishes to construct paths which will help in the study of the Laplacian and related operators, one uses $O(M)$, the bundle of orthonormal frames on M, as the manifold N, and considers the m canonical horizontal vector fields on this bundle as the vector fields A_a. Suppose that (p, e_a) is a point in $O(M)$, that is, p is a point in M and e_a, $a = 1, \ldots, m$ is an orthonormal basis of the tangent space at p. Then, if x^i are local coordinates at p and

$$e_a = e_a^i \frac{\partial}{\partial x^i} \qquad (3.9)$$

the m canonical horizontal vector fields on $O(M)$ are

$$V_a = e_a^i \frac{\partial}{\partial x^i} - e_a^i e_b^j \Gamma_{ij}{}^k(x) \frac{\partial}{\partial e_b^k} \qquad (3.10)$$

where the functions $\Gamma_{im}{}^k$ are the Christoffel symbols for the Riemannian connection. In this case, following the general form of (3.3), one obtains the stochastic differential equations

$$\mathrm{d}x_s^i = e_{a,s}^i \circ \mathrm{d}b_s^a \qquad \mathrm{d}e_{a,s}^i = -e_{a,s}^j e_{b,s}^k \Gamma_{jk}{}^i(x_s) \circ \mathrm{d}b_s^b\,. \qquad (3.11)$$

To see the connection with the Laplacian, suppose that x_s^i, $e_{a,s}^i$, $(a, i = 1, \ldots, m)$ are solutions to (3.11) with initial condition (in a particular coordinate system) $x_0 = x \in M$, $e_{a,0}^i = e_a^i$, where $e_a = e_a^i \partial/\partial x^i$ is an orthonormal frame at x. Then, again omitting details of the patching of solutions on different coordinate charts, if f is a smooth function on $O(M)$,

$$\mathbb{E}(f(x_t, e_t)) - f(x, e) = \int_0^t \frac{1}{2} V_a V_a f(x_s, e_s)\mathrm{d}s\,. \qquad (3.12)$$

In the particular case where f depends on x only, (that is, $f = g \circ \pi$ where $\pi : O(M) \to M$ is the projection map, and $g \in C^\infty(M)$),

$$\mathbb{E}\big(g(x_t)\big) - g(x) = \int_0^t -L_{\mathrm{scal}}\, g(x_s)\mathrm{d}s \tag{3.13}$$

where $L_{\mathrm{scal}} = -\frac{1}{2}(g^{ij}\partial_i\partial_j + g^{ij}\Gamma_{ij}{}^k\partial_k)$ is the scalar Laplacian. This follows since

$$\tfrac{1}{2}V_a V_a f(x) = -L_{\mathrm{scal}} f(x) \tag{3.14}$$

and one can show that almost certainly $e^i_{a,s}e^j_{a,s} = g^{ij}(x_s)$. Hence one finds that $f(x,t) = \mathbb{E}\big(g(x_t)\big)$ satisfies

$$\frac{\partial f}{\partial t} = -L_{\mathrm{scal}} f\,. \tag{3.15}$$

The two approaches to Brownian motion on Riemann manifolds are closely related, because the x components of the process which solves the stochastic differential equations (3.11) have finite-dimensional distributions built from the heat kernel of the Laplacian.

Turning now to supermanifolds, a construction analogous to this second method will be described. While more general supermanifolds are possible, attention in this paper will be restricted to those of the type constructed in section 2, because of their use in geometry and supersymmetric physics; the general approach is applicable to other situations. The Brownian paths in superspace which will now be defined will lead to a Feynman–Kac formula for the Laplace–Beltrami operator $L = (d+\delta)^2$ acting on $C^{\infty\prime}(S(E),\mathbb{C})$ introduced in section 2. Letting $(\theta^a_t, \rho^a_t) a = 1,\ldots,m$ denote m-dimensional fermionic Brownian paths [2] and letting $(x^i, e^i_a, \theta^i, \eta^r)$ be coordinates of a point in the extended bundle of orthognal frames $S(O(M),E)$ introduced in section 2, consider the $m + m^2 + n$ stochastic differential equations

$$\begin{aligned}
x^i_t &= x^i + \int_0^t e^i_{a,s}\circ \mathrm{d}b^a_s \qquad e^i_{a,t} = e^i_a + \int_0^t -e^\ell_{a,s}\Gamma_{k\ell}{}^i(x_s)e^k_{b,s}\circ\mathrm{d}b^b_s\\
\xi^i_t &= \theta^i + \theta^a_t e^i_{a,t} + \int_0^t\Big(-\xi^j_s\Gamma_{jk}{}^i(x_s)e^k_{b,s}\circ\mathrm{d}b^b_s - \theta^a_t\,\mathrm{d}e^i_{a,s}\\
&\qquad + \frac{\mathrm{i}}{4}\xi^j_s R^i{}_{jkl}(x_s)\xi^k_s\pi^l_s\mathrm{d}s\Big)\\
\eta^p_t &= \eta^p + \int_0^t\Big(-e^j_{a,s}\eta^q_s A_j{}_q{}^p(x_s)\circ\mathrm{d}b^a_s\\
&\qquad + \frac{1}{4}\eta^q_s(\xi^i_s + \mathrm{i}\pi^i_s)(\xi^j_s + \mathrm{i}\pi^j_s)F_{ij}{}_q{}^p(x_s)\mathrm{d}s\Big)
\end{aligned}\tag{3.16}$$

where

$$\pi^\ell_s = e^\ell_{a,s}\rho^a_s\,. \tag{3.17}$$

The existence of local solutions to such stochastic differential equations was established in theorem 5.2 of [2], while the usual patching techniques allow a global solution to be constructed, since they transform covariantly under change of coordinates.

In order to establish a Feynman–Kac formula for the Laplace–Beltrami operator $L = (d+\delta)^2$, vector fields $W_a (a = 1, \ldots, m)$ on $S(O(M), E)$ must be defined which are the canonical horizontal vector fields on $S(O(M), E)$ regarded as a bundle over M with connection (Γ, A). In a local coordinate system $(x^i, e^i_a, \theta^i, \eta^p)$ on $S(O(M), E)$ these vector fields take the form

$$W_a = e^i_a \frac{\partial}{\partial x_i} - e^j_a e^k_b \Gamma_{jk}{}^i \frac{\partial}{\partial e^i_b} - e^j_a \theta^k \Gamma_{jk}{}^i \frac{\partial}{\partial \theta^i} - e^j_a \eta^r A_j{}_r{}^s \frac{\partial}{\partial \eta^s}. \tag{3.18}$$

The key property of the vector fields W_a is that, when acting on functions on $S(O(M), E)$ which are independent of the e^i_a (that is, on functions of the form $f = g \circ \pi$ where π is the canonical projection of $S(O(M), E)$ onto $S(E)$) it is related to the Laplace–Beltrami operator $L = \frac{1}{2}(d+\delta)^2$ by

$$-\tfrac{1}{2}\left(W_a W_a - R^j_i(x)\theta^i \delta_{\theta^j} - \tfrac{1}{2} R_{ki}{}^{j\ell}(x)\theta^i\theta^k \delta_{\theta^j}\delta_{\theta^\ell} + \tfrac{1}{4}[\psi^i, \psi^j] F_{ij}{}_r{}^s(x)\eta^r \delta_{\eta^s}\right) = L \tag{3.19}$$

as may easily be seen from lemma (2.1). The following Feynman–Kac formula may then be established quite directly using theorem (3.2).

Theorem 3.3. Suppose that $(x^i_t, e^i_{a,t}, \eta^r_t)$ satisfy (3.16). Then

$$\exp(-Lt) g(x, \theta, \eta) = \mathbb{E}\big(g(x_t, \xi_t, \eta_t)\big) \tag{3.20}$$

where $L = \frac{1}{2}(d+\delta)^2$ is the Laplace–Beltrami operator acting on $C^{\infty\prime}(S(E), \mathbb{C})$.

Proof. Using [13] and the superspace Itô formula (theorem 3.5 of [2]),

$$\begin{aligned}
&\mathbb{E}\big(g(x_t, \xi_t, \eta_t)\big) - g(x, \theta, \eta) \\
&\quad = \mathbb{E}\Big(\int_0^t \frac{1}{2} W_a W_a g(x_s, \xi_s, \eta_s) - \frac{\mathrm{i}}{4} R^i{}_{jk\ell}(x_s)\xi^k_s \pi^\ell_s \xi^j_s \delta_{\theta^i} g(x_s, \xi_s, \eta_s) \\
&\qquad + \tfrac{1}{4}(\xi^i_s + i\pi^i_s)(\xi^j_s + i\pi^j_s) F_{ij}{}_p{}^q \eta^p_t \delta_{\eta^q} g(x_s, \xi_s, \eta_s)\,\mathrm{d}s\Big) \\
&\quad = \mathbb{E}\Big(\int_0^t \frac{1}{2}\big(W_a W_a - R^j_i(x)\theta^i \delta_{\theta^j} - \frac{1}{2} R_{ki}{}^{j\ell}(x)\theta^i\theta^k \delta_{\theta^j}\delta_{\theta^\ell} \\
&\qquad + \tfrac{1}{4}[\psi^i, \psi^j] F_{ij}{}_r{}^s(x)\eta^r \delta_{\eta^s}\big) g(x_s, \xi_s, \eta_s)\,\mathrm{d}s\Big)
\end{aligned} \tag{3.21}$$

using properties of fermionic paths ([2] equation (2.15)). Hence, if

$$\begin{aligned}
f(x, \theta, \eta, t) &= \mathbb{E}\big(g(x_t, \xi_t, \eta_t)\big) \\
f(x, \theta, \eta, t) - f(x, \theta, \eta, 0) &= \int_0^t -L f(x, \theta, \eta, s)\,\mathrm{d}s
\end{aligned} \tag{3.22}$$

and the result follows. □

4. The Atiyah–Singer index theorem

The supermanifold stochastic techniques developed in the previous section will now be used to establish the Atiyah–Singer index theorem. As in the paper by Atiyah [24], the method used is to establish a stronger local version of the theorem in the particular case of the twisted Hirzebruch signature complex, by studying the heat kernel of the Laplacian. The full theorem may then be inferred by the K-theoretic arguments presented in [24].

The starting point is the formula of McKean and Singer [17] and Witten [18] expressing the index of the complex in terms of the heat kernel of the Laplacian. As before, suppose that (M, g) is a compact Riemannian manifold of dimension $m = 2k$, and that E is an n-dimensional Hermitian vector bundle over M. The formula of McKean and Singer then states that

$$\mathrm{Index}\ (d+\delta) = \mathrm{Str}\exp(-Lt)\,. \qquad (4.1)$$

Here $d+\delta$ is, as before, the Hodge–de Rham operator and $L = \frac{1}{2}(d+\delta)^2$ is the Laplace–Beltrami operator, while Str denotes the supertrace. With the identification of the space of twisted forms on M with the space of functions $C^{\infty\prime}(S(E),\mathbb{C})$ on the supermanifold $S(E)$ set up in section 2, the supertrace can be defined in the following manner. First, the standard involution τ may be defined on $C^{\infty\prime}(S(E),\mathbb{C})$ by the formula

$$\tau\Big(\sum_{\mu\in M_m}\sum_{r=1}^{n} f_{\mu r}(x)\theta^\mu\eta^r\Big) = \sum_{\mu\in M_m}\sum_{r=1}^{n}\int_{\mathcal{B}} \mathrm{d}^m\rho \frac{1}{\sqrt{\det(g_{ij}(x))}} \exp \mathrm{i}\rho^i\theta^j g_{ij}(x)\ f_{\mu r}(x)\rho^\mu\eta^r\,. \qquad (4.2)$$

(In this expression $\rho^1, \ldots, \rho^m$ are anticommuting variables and the integral is the Berezin integral defined in section 2. The definition is independent of the choice of local coordinates.) The supertrace is now defined for a suitable operator O on $C^{\infty\prime}(S(E),\mathbb{C})$ by the formula

$$\mathrm{Str}\, O = \mathrm{Tr}\, \tau O \qquad (4.3)$$

where Tr denotes the conventional trace. It emerges in the proof of McKean and Singer's formula that the right-hand side of (4.1) is in fact independent of the real parameter t.

The Atiyah–Singer index theorem for the twisted Hirzebruch signature complex takes the following form.

Theorem 4.1. With the notation of section 2,

$$\mathrm{Index}\,(d+\delta) = \int_M \left[\mathrm{tr}\ \exp\Big(\frac{-F}{2\pi}\Big) \det\Big(\frac{\mathrm{i}\Omega/2\pi}{\tanh \mathrm{i}\Omega/2\pi}\Big)^{\frac{1}{2}}\right] \qquad (4.4)$$

where F is the curvature 2-form of a connection on the bundle E, Ω is the Riemann curvature 2-form of (M, g) and the square brackets indicate projection onto the m-form component of the integrand.

Combining this with the McKean and Singer formula (4.1) one sees that an equivalent result is

$$\operatorname{Str}\exp(-Ht) = \int_M \left[\operatorname{tr}\,\exp\left(\frac{-F}{2\pi}\right)\det\left(\frac{\mathrm{i}\Omega/2\pi}{\tanh \mathrm{i}\Omega/2\pi}\right)^{\frac{1}{2}}\right]. \tag{4.5}$$

(Again this result is valid for all t.)

The following theorem is a stronger, local version of this result.

Theorem 4.2. With the notation of theorem 4.1, if $p \in M$,

$$\lim_{t\to 0} \operatorname{str}\exp(-Ht)(p,p)\,\mathrm{dvol} = \left[\operatorname{tr}\,\exp\left(\frac{-F}{2\pi}\right)\det\left(\frac{\mathrm{i}\Omega/2\pi}{\tanh \mathrm{i}\Omega/2\pi}\right)^{\frac{1}{2}}\right]\Bigg|_p \tag{4.6}$$

where str denotes the $2^m n \times 2^m n$ matrix supertrace, as opposed to the full operator supertrace Str, so that $\operatorname{str}\exp(-Ht)(p,q)$ is then the kernel of the operator on $C^\infty(M)$ obtained by this partial supertrace.

The strategy for proving this theorem is to use the Feynman–Kac formula (theorem (3.3)) to analyse the operator $\exp(-Ht)$ and then, using Duhamel's formula to extract information about the kernel (as in Getzler [25]), show that in the limit as t tends to zero only the required terms survive. The proof is thus carried out in several steps.

Step 1. An expression for the matrix supertrace in terms of Berezin integrals:

As a peliminary, the matrix supertrace of an operator will be expressed in terms of a Berezin integral of its kernel. Suppose that $G(p)$ denotes the 2^p-dimensional space of polynomial functions of p anticommuting variables; then a linear operator O on this space has a kernel $O(\xi,\zeta) = O(\xi_1,\ldots,\xi_p,\ \zeta_1,\ldots,\zeta_p)$ which is a function of $2p$ anticommuting variables, and satisfies

$$O\,f(\xi) = \int_B \mathrm{d}^p\zeta\, O(\xi,\zeta) f(\zeta) \tag{4.7}$$

for all functions f in $G(p)$. It can be shown by direct calculation that

$$\operatorname{tr} O = \int_B \mathrm{d}^p\xi\, O(\xi,-\xi)\,. \tag{4.8}$$

Thus if O is an operator on $C^{\infty\prime}(S(E),\mathbb{C})$ one has the local coordinate expression

$$\begin{aligned}\operatorname{str} O(x,y) &= \operatorname{tr}\tau O(x,y)\\ &= \int_B \mathrm{d}^m\rho\,\mathrm{d}^m\theta\,\mathrm{d}^n\eta\, O(x,y,\rho,-\theta,\eta,-\eta)\\ &\quad\times \frac{1}{\sqrt{\det(g_{ij}(x))}}\exp \mathrm{i}\rho^i\theta^j g_{ij}(x)\,.\end{aligned} \tag{4.9}$$

Step 2. The construction of a locally equivalent metric and connection on the super extension of $\mathbb{R}^m \times \mathbb{C}$:

Theorem (4.2) is a local result; in order to prove this theorem at an arbitary but fixed point $p \in M$ it is in fact sufficient to replace the manifold M by $\mathbb{R}^m$ and the bundle E over M by the trivial bundle $\mathbb{R}^m \times \mathbb{C}^n$ over $\mathbb{R}^m$ with metric and connection satisfying certain conditions, and to prove the result for this simpler situation. The construction of a suitable metric and connection will now be given. Suppose that W is an open subset of M containing p which has compact closure and is both a coordinate neighbourhood of M and a local trivialization neighbourhood of the bundle E, and that U is also an open subset of M containing p with $\overline{U} \subset W$. Also let $\phi : W \to \mathbb{R}^m$ be a system of normal coordinates on W based at p which satisfy $\det g_{ij}(x) = 1$ at all points of U (and, of course, $\phi(p) = 0$) [21]. Additionally a local trivialization of the bundle E is chosen such that $A_{i\,r}{}^{s}(0) = 0$. Then the required metric $\tilde{g}$ on $\mathbb{R}^m$ is a metric satisfying

$$\begin{aligned}\tilde{g}_{ij}(x^1,\ldots,x^m) &= g_{ij}(\phi^{-1}(x^1,\ldots,x^m)) \qquad \text{when} \quad x \in \phi(U)\\ \tilde{g}_{ij}(x^1,\ldots,x^m) &= \delta_{ij} \qquad \text{when} \quad x \notin \phi(W)\end{aligned} \tag{4.10}$$

and $\det \tilde{g}_{ij} = 1$ throughout $\mathbb{R}^m$.

Also, the required connection is a connection satisfying

$$\begin{aligned}\tilde{A}_{i\,r}{}^{s}(x^1,\ldots,x^m) &= A_{i\,r}{}^{s}(\phi^{-1}(x^1,\ldots,x^m)) \qquad \text{when} \quad x \in \phi(U)\\ \tilde{A}_{i\,r}{}^{s}(x^1,\ldots,x^m) &= 0 \qquad \text{when} \quad x \notin \phi(W)\,.\end{aligned} \tag{4.11}$$

Some simple consequences of this definition are that, if $R_{ijk}{}^{l}$ denotes the Riemann curvature of (M,g) and $F_{ij\,r}{}^{s}$ denotes the curvature of the connection A on the bundle E, while tildes denote the corresponding quantities on $\mathbb{R}^m$ and $\mathbb{R}^m \times \mathbb{C}^n$,

$$\begin{aligned}\tilde{R}_{ijk}{}^{l}(x^1,\ldots,x^m) &= R_{ijk}{}^{l}(\phi^{-1}(x^1,\ldots,x^m))\\ \tilde{F}_{ij\,r}{}^{s}(x^1,\ldots,x^m) &= F_{ij\,r}{}^{s}(\phi^{-1}(x^1,\ldots,x^m))\end{aligned} \tag{4.12}$$

on $\phi(U)$. Also one has the standard Taylor expansions in normal coordinates [24]

$$\begin{aligned}\tilde{g}_{ij}(x) &= \delta_{ij} - \tfrac{1}{3}x^k x^\ell \tilde{R}_{ki\ell j}(0) + \cdots\\ \tilde{\Gamma}_{ij}{}^{k}(x) &= \tfrac{1}{3}x^\ell(\tilde{R}_{lji}{}^{k}(0) + \tilde{R}_{lij}{}^{k}(0)) + \cdots\\ \tilde{A}_{i\,r}{}^{s}(x) &= -\tfrac{1}{2}x^j \tilde{F}_{ij\,r}{}^{s}(0) + \cdots\,.\end{aligned} \tag{4.13}$$

Cutting and pasting arguments, for instance as presented by Cycon *et al* in [21], shows that (if $\tilde{H}$ denotes the Laplacian $\frac{1}{2}(d+\delta)^2$ on $S(\mathbb{R}^m \times \mathbb{C}^n)$ with metric $\tilde{g}$ and connection $\tilde{A}$)

$$\lim_{t\to 0}(\operatorname{str}\exp(-Ht)(p,p) - \operatorname{str}\exp(-\tilde{H}t(0,0)) = 0\,. \tag{4.14}$$

Thus in the rest of this section it will be sufficient to consider $\tilde{H}$ on $S(\mathbb{R}^m \times \mathbb{C}^n)$ in place of H on $S(E)$.

Step 3. The use of the Feynman–Kac formula to give an explicit expression for the required matrix supertrace:

Letting $\tilde{H}^0$ be the flat Laplacian

$$\tilde{H}^0 = -\tfrac{1}{2}\partial_i\partial_i \tag{4.15}$$

on $S(\mathbb{R}^m \times \mathbb{C}^n)$, and $\tilde{K}_t(x,x',\theta,\theta',\eta,\eta')$ and $\tilde{K}_t^0(x,x',\theta,\theta',\eta,\eta')$ denote the heat kernels $\exp -\tilde{H}t(x,x',\theta,\theta',\eta,\eta')$ and $\exp -\tilde{H}^0 t(x,x',\theta,\theta',\eta,\eta')$, Duhamel's formula [25] states that

$$\begin{aligned}\tilde{K}_t(x,x',\theta,\theta',\eta,\eta') &- \tilde{K}_t^0(x,x',\theta,\theta',\eta,\eta')\\ &= \int_0^t \mathrm{d}s\,\mathrm{e}^{-(t-s)\tilde{H}}(\tilde{H}-\tilde{H}^0)\tilde{K}_s^0(x,x',\theta,\theta',\eta,\eta')\end{aligned} \tag{4.16}$$

where all differential operators act with respect to the variables x,θ and η. Now

$$\begin{aligned}\tilde{K}_s^0(x,x',\theta,\theta',\eta,\eta')\\ &= \int_B \mathrm{d}^m\rho\,\mathrm{d}^n\kappa(2\pi s)^{-m/2}\exp -[(x-x')^2/2s]\\ &\quad\times \exp[-\mathrm{i}\rho(\theta-\theta')]\exp[-\mathrm{i}\kappa(\eta-\eta')]\end{aligned} \tag{4.17}$$

and direct calculation shows that $\operatorname{str}\tilde{K}_t^0(x,x') = 0$. Thus, using (4.9),

$$\begin{aligned}\operatorname{str}\tilde{K}_t(0,0) = &\int_B \mathrm{d}^m\theta\,\mathrm{d}^m\theta'\,\mathrm{d}^n\eta\\ &\times\int_0^t \mathrm{d}s\{(\mathrm{e}^{-(t-s)\tilde{H}}(\tilde{H}-\tilde{H}^0)\tilde{K}_s^0(0,0,\theta,\theta',\eta,\eta'))\big|_{\eta'=-\eta}\\ &\times\exp[-\mathrm{i}\theta\theta']\}\,.\end{aligned} \tag{4.18}$$

Now, using the Feynman–Kac formula (3.20),

$$\begin{aligned}\mathrm{e}^{-(t-s)\tilde{H}}&(\tilde{H}-\tilde{H}^0)\tilde{K}_s^0(0,0,\theta,\theta',\eta,\eta')\\ &= \mathbb{E}\int_B \mathrm{d}^m\rho\,\mathrm{d}^n\kappa(2\pi s)^{-m/2}\\ &\quad\times F_s(\tilde{x}_{t-s},\tilde{\xi}_{t-s},\rho,\tilde{\eta}_{t-s},\kappa)\exp -\mathrm{i}\rho\theta'\exp\mathrm{i}\kappa\eta'\end{aligned} \tag{4.19}$$

where

$$F_s(x,\theta,\rho,\eta,\kappa) = (\tilde{H}-\tilde{H}^0)([\exp -(x^2/2s)\exp(-\mathrm{i}\rho\theta)\exp(-\mathrm{i}\kappa\eta)] \tag{4.20}$$

(with differential operators again acting with respect to x, θ and η) and $\tilde{x}_s$, $\tilde{\xi}_s$ and $\tilde{\eta}_s$ satisfy the stochastic differential equation (3.16) (in normal coordinates on $S(\mathbb{R}^m \times \mathbb{C}^n)$) with initial conditions $\tilde{x}_0 = 0$, $\tilde{e}^i_{a\,0} = \delta^i_a$, $\tilde{\xi}_0 = \theta$ and $\tilde{\eta}_0 = \eta$. Hence

$$\begin{aligned}\operatorname{str}\tilde{K}_t(0,0) = \mathbb{E}\bigg[&\int_B \mathrm{d}^m\theta\,\mathrm{d}^n\theta'\,\mathrm{d}^m\rho\,\mathrm{d}^n\eta\,\mathrm{d}^n\kappa\\ &\times\int_0^t \mathrm{d}s(2\pi s)^{-m/2}F_s(x_{t-s},\xi_{t-s},\rho,\eta_{t-s},\kappa)\\ &\times\exp(-\mathrm{i}\theta\theta')\exp(\mathrm{i}\rho\theta')\exp(-\mathrm{i}\kappa\eta)\bigg]\,.\end{aligned} \tag{4.21}$$

Thus, integrating out θ' and ρ, one obtains

$$\operatorname{str} \tilde{K}_t(0,0) = \mathbb{E}\int_B \mathrm{d}^m\theta \mathrm{d}^n\eta \mathrm{d}^n\kappa \int_0^t \mathrm{d}s(2\pi s)^{-m/2} F_s(x_{t-s}, \xi_{t-s}, \theta, \eta_{t-s}, \kappa) \times \exp(-\mathrm{i}\kappa\eta)\,. \tag{4.22}$$

Step 4. The replacement of the Hamiltonian $\tilde{H}$ by an equivalent Hamiltonian $\tilde{H}^1$ of simpler form:

The next step is to construct a simpler Hamiltonian $\tilde{H}^1$ on $S(\mathbb{R}^m \times \mathbb{C}^n)$with heat kernel $\tilde{K}^1{}_t(x, x', \theta, \theta', \eta, \eta')$ such that

$$\lim_{t\to 0} \operatorname{str} \tilde{K}^1{}_t(0,0) = \lim_{t\to 0} \operatorname{str} \tilde{K}_t(0,0) \tag{4.23}$$

so that the required supertrace can be calculated. The modified Hamiltonian $\tilde{H}^1$ is obtained by considering the following stochastic differential equations (which are a simplification of (3.16)):

$$\begin{aligned}
\tilde{x}^1{}^i_t &= b^i_t \\
\tilde{\xi}^1{}^i_t &= \theta^i + \theta^a_t \delta^i_a + \int_0^t \Big(\frac{1}{3}\tilde{\xi}^1{}^j_s \tilde{x}^1{}^\ell_s (R_{\ell kj}{}^i + R_{\ell jk}{}^i)\mathrm{d}b^k_s \\
&\qquad + \frac{1}{3}\tilde{\xi}^1{}^j_s R^i_j \mathrm{d}s - \frac{\mathrm{i}}{4}\tilde{\xi}^1{}^j_s \tilde{\xi}^1{}^k_s \tilde{\pi}^1{}^\ell_s R_{jk}{}^i{}_l \,\mathrm{d}s\Big) \\
\tilde{\eta}^1{}^p_t &= \eta^p + \int_0^t \frac{1}{4}\tilde{\eta}^1{}^q_s (\theta^i_s - \mathrm{i}\pi^i_s)(\theta^j_s - \mathrm{i}\pi^j_s) F_{ij\,q}{}^p \mathrm{d}s\,.
\end{aligned} \tag{4.24}$$

with

$$\tilde{\pi}^1{}^l_s = \rho^a_s \delta^\ell_a. \tag{4.25}$$

(Here $R_{ijkl} = \tilde{R}^1{}_{ijkl}(0)$ and $F_{ij\,p}{}^q = \tilde{F}^1{}_{ij\,p}{}^q(0)$, with indices raised and lowered by $\tilde{g}^1_{ij}(0) = \delta_{ij}$.)

Then, if $f \in C^{\infty\prime}(S(\mathbb{R}^m \times \mathbb{C}^n))$,

$$\mathbb{E}\big(f(\tilde{x}^1{}_t, \tilde{\xi}^1{}_t, \tilde{\eta}^1{}_t)\big) - f(0,\theta,\eta) = \mathbb{E}\Big(\int_0^t -\tilde{H}^1 f(\tilde{x}^1{}_s, \tilde{\xi}^1{}_s, \tilde{\eta}^1{}_s)\mathrm{d}s\Big) \tag{4.26}$$

where

$$\begin{aligned}
\tilde{H}^1 = -\big[&\tfrac{1}{2}\partial_i\partial_i - \tfrac{1}{4}R_{ij}{}^{kn'}\hat{\theta}^i\hat{\theta}^j\delta_{\theta^{n'}}\delta_{\theta^k} + \tfrac{1}{3}R^k_j\hat{\theta}^j\delta_{\theta^k} \\
&- \tfrac{1}{2}R^k_j\hat{\theta}^j\delta_{\theta^k} + \tfrac{1}{4}\psi^i\psi^j\hat{\eta}^q\tilde{F}^1{}_{ij\,p}{}^q\delta_{\eta^q} - \tfrac{1}{3}\hat{\theta}^j\hat{x}^\ell(R_\ell{}^k{}_j{}^i + R_{\ell j}{}^{ki})\delta_{\theta^i}\partial_k \\
&- \tfrac{1}{18}\hat{\theta}^{n'}\hat{\theta}^{m'}\hat{x}^k\hat{x}^l(R_{kpn'}{}^i + R_{kn'p}{}^i)(R_\ell{}^p{}_{m'}{}^j + R^{pj}_{\ell m'})\delta_{\theta^i}\delta_{\theta^j}\big]\,.
\end{aligned} \tag{4.27}$$

(Here the physicist's convention of denoting operators with hats is adopted to avoid ambiguities.)

Now, using Duhamel's formula again, and setting

$$G_s(x,\xi,\theta,\eta,\kappa) = \exp -(x^2/2s + \mathrm{i}\theta\xi + \mathrm{i}\kappa\eta) \tag{4.28}$$

one finds that

$$\begin{aligned}\tilde{K}_t(0,0) - \tilde{K}^1{}_t(0,0) &= \mathbb{E}\int_{\mathcal{B}} \mathrm{d}^m\theta\,\mathrm{d}^n\eta\,\mathrm{d}^n\kappa \int_0^t \mathrm{d}s\,(2\pi s)^{-m/2} \\ &\quad\times\Big[(\tilde{H} - \tilde{H}^0)G_s(x,\xi,\theta,\eta,\kappa)\Big|_{x=\tilde{x}_{t-s},\xi=\tilde{\xi}_{t-s},\eta=\tilde{\eta}_{t-s}} \\ &\quad - (\tilde{H}^1 - \tilde{H}^0)G_s(x,\xi,\theta,\eta,\kappa)\Big|_{x=\tilde{x}^1{}_{t-s},\xi=\tilde{\xi}^1{}_{t-s},\eta=\tilde{\eta}^1{}_{t-s}}\Big]. \end{aligned}\tag{4.29}$$

At this stage it is useful to introduce the scaled variable $\phi = \sqrt{2\pi t}\theta$; then, following the rules of Berezin integration [22], $\mathrm{d}\theta = \sqrt{2\pi t}\,\mathrm{d}\phi$ and thus

$$\tilde{K}_t(0,0) - \tilde{K}^1{}_t(0,0) = A(t) + B(t)$$

where

$$\begin{aligned}A(t) &= \mathbb{E}\int_{\mathcal{B}} \mathrm{d}^m\phi\,\mathrm{d}^n\eta\,\mathrm{d}^n\kappa\,(2\pi t)^{m/2} \int_0^t \mathrm{d}s\,(2\pi s)^{-m/2} \\ &\quad\times\left[(\tilde{H} - \tilde{H}^1)G_s\left(x,\xi,\frac{\phi}{\sqrt{2\pi t}},\eta,\kappa\right)\Big|_{x=\tilde{x}^1{}_{t-s},\xi=\tilde{\xi}^1{}_{t-s},\eta=\tilde{\eta}^1{}_{t-s}}\right]\end{aligned}\tag{4.30}$$

and

$$\begin{aligned}B &= \mathbb{E}\int_{\mathcal{B}} \mathrm{d}^m\phi\,\mathrm{d}^n\eta\,\mathrm{d}^n\kappa\,(2\pi t)^{m/2} \int_0^t \mathrm{d}s\,(2\pi s)^{-m/2} \\ &\quad\times\Big[(\tilde{H} - \tilde{H}^0)G_s(x,\xi,\frac{\phi}{\sqrt{2\pi t}},\eta,\kappa)\Big|_{x=\tilde{x}_{t-s},\xi=\tilde{\xi}_{t-s},\eta=\tilde{\eta}_{t-s}} \\ &\quad - (\tilde{H} - \tilde{H}^0)G_s(x,\xi,\frac{\phi}{\sqrt{2\pi t}},\eta,\kappa)\Big|_{x=\tilde{x}^1{}_{t-s},\xi=\tilde{\xi}^1{}_{t-s},\eta=\tilde{\eta}^1{}_{t-s}}\Big].\end{aligned}$$

Now, using flat space Brownian motion techniques [26] together with fermionic Brownian motion techniques [1], one can show that for any suitably regular function f on $S(\mathbb{R}^m \times \mathbb{C}^n)$,

$$\begin{aligned}\mathbb{E}f(\tilde{x}^1{}_{t-s},\tilde{\xi}^1{}_{t-s},\tilde{\eta}^1{}_{t-s}) &= \exp[-\tilde{H}^1(t-s)]f(x,\frac{\phi}{\sqrt{2\pi t}},\eta) \qquad \text{(by 4.26)} \\ &= \mathbb{E}\Big[\exp\Big(\int_0^{t-s} -\frac{\mathrm{i}}{4}(\theta^i_u - \mathrm{i}\pi^i_u)(\theta^j_u - \mathrm{i}\pi^j_u)\tilde{F}^1{}_{ij\,p}{}^q\eta^{0\,p}_u\kappa^0_{q\,u}\mathrm{d}u \\ &\quad -\frac{1}{4}R_{ijkl}\theta^i_u\theta^j_u\rho^k_u\rho^l_u\,\mathrm{d}u + \frac{\mathrm{i}}{3}R_{ij}\theta^i_u\rho^j_u\mathrm{d}u - \frac{\mathrm{i}}{3}\theta^j_u\rho^k_u b^\ell_u(R_{\ell kj}{}^i + R_{\ell jk}{}^i)\mathrm{d}b^k_u\Big) \\ &\quad\times f\left(b_{t-s},\frac{\phi}{\sqrt{2\pi t}} + \theta_{t-s},\eta + \eta^0_{t-s}\right)\Big].\end{aligned}\tag{4.31}$$

(Here $\theta_u^i, \rho_u^j, i, j = 1, \ldots, m$ are fermionic Brownian paths, while $\eta_u^{0\,p}, \kappa^0_{q\,u}$ are a further set of fermionic Brownian paths, introduced to handle the twisted or gauge group part of the operator.) Thus $A(t)$ can be estimated using $b_u \sim \sqrt{u}$, $\theta_u \sim 1, \rho_u \sim 1, \eta_u^0 \sim 1, \kappa_u^0 \sim 1$. (The fermionic part of these estimates follows from theorem 3.3 of [2].) This estimation shows that $A(t) \to 0$ as $t \to 0$.

Now the stochastic differential equations (3.16) (in the normal coordinate system) and (4.24) are closely related, and one might expect that for small s their solutions would be similar. In fact, using the explicit construction of solutions developed in [2], one can show (by induction) that $\tilde{x}_u - \tilde{x}^1{}_u \sim \sqrt{u^3}$, $\tilde{\xi}_u - \tilde{\xi}^1{}_u \sim u$ and $\tilde{\eta}_u - \tilde{\eta}^1{}_u \sim u$ and $\tilde{e}_u - \tilde{e}_u^1 \sim u$. This enables one to show that $B(t) \to 0$ as $t \to 0$. Thus

$$\lim_{t\to 0} \operatorname{str}\exp[-\tilde{H}t(0,0)] = \lim_{t\to 0} \operatorname{str}\exp[-\tilde{H}^1 t(0,0)]\,. \tag{4.32}$$

Step 5. Evaluating the supertrace:

The final step in the proof of the Atiyah–Singer index theorem is to evaluate $\lim_{t\to 0} \operatorname{str}\exp[-\tilde{H}^1 t(0,0)]$ using flat-space path-integral techniques (both classical [21] and fermionic [1]).

Now, once again using Duhamel's formula, and also using (4.31),

$$\begin{aligned}
&\operatorname{str}\exp[-\tilde{H}^1 t(0,0)] \\
&\quad= \mathbb{E}\Bigg[\!\!\Bigg[\int_{\mathcal{B}} \mathrm{d}^m\phi\, \mathrm{d}^n\eta\, \mathrm{d}^n\kappa\, (2\pi t)^{m/2} \\
&\qquad\times \int_0^t \mathrm{d}s\, (2\pi s)^{-m/2} \mathrm{d}^m\theta' \bigg\{\exp[-\tilde{H}^1(t-s)] \\
&\qquad\times (\tilde{H}^1 - \tilde{H}^0) G_s\left(0, \frac{\phi}{\sqrt{2\pi t}}, \theta', \eta, -\eta\right) \exp -(\mathrm{i}\kappa\eta) \exp\left(-\mathrm{i}\frac{\phi}{\sqrt{2\pi t}}\theta'\right)\bigg\}\Bigg]\!\!\Bigg] \\
&\quad= \mathbb{E}\left[\int_{\mathcal{B}} \mathrm{d}^m\theta\, \mathrm{d}^n\eta\, \mathrm{d}^n\kappa \int_0^t \mathrm{d}s\, (2\pi s)^{-m/2}\right. \\
&\qquad\times \left[\exp\left(\int_0^{t-s} \frac{1}{4}(\theta_u^i - \mathrm{i}\pi_u^i)(\theta_u^j - \mathrm{i}\pi_u^j) F_{ij\,p}{}^q (\mathrm{i}\eta_u^{0\,q}\kappa^0_{p\,u} - \delta_p^q)\mathrm{d}u\right.\right. \\
&\qquad - \frac{1}{4} R_{ijkl}\theta_u^i\theta_u^j\rho_u^k\rho_u^l + \frac{\mathrm{i}}{3} R_{ij}\theta_u^i\rho_u^j \mathrm{d}u + \frac{\mathrm{i}}{3}\theta_u^j\rho_u^k b_u^\ell (R_{\ell kj}{}^i + R_{\ell jk}{}^i)\mathrm{d}b_u^k\bigg) \\
&\qquad\times (\tilde{H}^1 - \tilde{H}^0) G_s\left(x, \xi, \frac{\phi}{\sqrt{2\pi t}}, \alpha, \kappa\right)\bigg|_{x=b_{t-s}, \xi=\theta_{t-s}, \alpha=\eta^0_{t-s}} \exp \mathrm{i}\kappa\eta\bigg]\,.
\end{aligned} \tag{4.33}$$

Also, again using the estimates for flat space Brownian paths, it can be seen that

$$\begin{aligned}
&\lim_{t\to 0} \operatorname{str}\exp -\tilde{H}^1 t(0,0) \\
&\quad= \mathbb{E}\bigg\{\lim_{t\to 0}\int_{\mathcal{B}} \mathrm{d}^m\phi\, \mathrm{d}^n\eta\, \mathrm{d}^n\kappa\, (2\pi t)^{m/2} \int_0^t \mathrm{d}s\, (2\pi s)^{-m/2}\Big[(\tilde{H}^2 - \tilde{H}^0) \\
&\qquad\times G_s(x, \xi, \frac{\phi}{\sqrt{2\pi t}}, \alpha, \kappa)\Big|_{x=b_{t-s}, \xi=\theta_{t-s}, \alpha=\eta^0_{t-s}} \exp(-\mathrm{i}\kappa\eta)\Big]\bigg\}
\end{aligned} \tag{4.34}$$

where

$$\tilde{H}^2 = -\Big(\frac{1}{2}\partial_i\partial_i - \frac{1}{3}\frac{\phi^i}{\sqrt{2\pi t}}\frac{\phi^j}{\sqrt{2\pi t}}(R_\ell{}^k{}_{ji} + R_{\ell j}{}^k{}_i)\hat{x}^l\partial_k - \frac{1}{18}\hat{x}^k\hat{x}^\ell \times \frac{\phi^{n'}}{\sqrt{2\pi t}}\frac{\phi^{m'}}{\sqrt{2\pi t}}\frac{\phi^i}{\sqrt{2\pi t}}\frac{\phi^j}{\sqrt{2\pi t}}(R_{kpn'i} + R_{kn'pi})(R_\ell{}^p{}_{n'j} + R_{\ell m'}{}^p{}_j) + \frac{1}{4}R_{ijk\ell}\frac{\phi^i}{\sqrt{2\pi t}}\frac{\phi^j}{\sqrt{2\pi t}}\hat{\chi}^k\hat{\chi}^\ell + \frac{\phi^i}{\sqrt{2\pi t}}\frac{\phi^j}{\sqrt{2\pi t}}F_{ij\,p}{}^q\hat{\eta}^p\delta_{\eta^q}\Big) \tag{4.35}$$

(with $\hat{\chi}^i = \theta^i + \partial/\partial\theta^i$). Thus

$$\lim_{t\to 0}\operatorname{str}\exp -\tilde{H}^1 t(0,0) = \lim_{t\to 0}\operatorname{str}\exp -\tilde{H}^2 t(0,0)\,. \tag{4.36}$$

Now $\tilde{H}^2$ decouples into operators acting separately on the x variables, the θ variables and the η variables. Explicitly,

$$\tilde{H}^2 = \tilde{H}^2_x + \tilde{H}^2_\theta + \tilde{H}^2_\eta \tag{4.37}$$

where (after some use of the symmetry properties of R_{ijkl})

$$\tilde{H}^2_x = -\Big(\frac{1}{2}\partial_i\partial_i + \frac{1}{2}\hat{x}^\ell\frac{\phi^j}{\sqrt{2\pi t}}\frac{\phi^i}{\sqrt{2\pi t}}R_{ji\ell}{}^k\partial_k + \frac{1}{8}\hat{x}^k\hat{x}^\ell\frac{\phi^i\phi^j\phi^{n'}\phi^{m'}}{(2\pi t)^2}R_{n'ikp}R_{m'j\ell}{}^p\Big)$$
$$\tilde{H}^2_\theta = -\frac{1}{4}R_{ijk\ell}\frac{\phi^i\phi^j}{2\pi t}\hat{\chi}^k\hat{\chi}^\ell \qquad \tilde{H}^2_\eta = -\frac{\phi^i}{\sqrt{2\pi t}}\frac{\phi^j}{\sqrt{2\pi t}}F_{ij\,p}{}^q\hat{\eta}^p\delta_{\eta^q}\,. \tag{4.38}$$

Now $\exp[-\tilde{H}^2_x t(0,0)]$ can be evaluated using the result given by Simon [21] for $\mathbb{R}^2$ that, if

$$L = -\frac{1}{2}\partial_i\partial_i + \frac{\mathrm{i}B}{2}(x^1\partial_2 - x^2\partial_1) + \frac{1}{8}B^2[(x^1)^2 + (x^2)^2] \tag{4.39}$$

then

$$\exp[-Lt(0,0)] = \frac{B}{4\pi\sinh(\frac{1}{2}Bt)}. \tag{4.40}$$

Thus, if $\Omega_k{}^l = \frac{1}{2}\phi^i\phi^j R_{ijk}{}^l$ is regarded as an $m\times m$ matrix, skew-diagonalized as

$$(\Omega_k{}^l) = \begin{pmatrix} 0 & \Omega_1 & \dots & 0 & 0 \\ -\Omega_1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & \Omega_{\frac{1}{2}m} \\ 0 & 0 & \dots & -\Omega_{\frac{1}{2}m} & 0 \end{pmatrix} \tag{4.41}$$

$$\exp[-\tilde{H}^2_x t(0,0)] = \prod_{k=1}^{m/2}\frac{\mathrm{i}\Omega_k}{2\pi t}\,\frac{1}{\sinh(\mathrm{i}\Omega_k/2\pi)}\,. \tag{4.42}$$

Also, using fermion paths [1] or direct calculation,

$$\exp[-t\tilde{H}_\theta^2(\theta,\theta')] = \int_B \mathrm{d}^m\rho \Big\{ \exp[-\mathrm{i}\rho(\theta-\theta')] \times \prod_{k=1}^{m/2} \left[\cosh\frac{\mathrm{i}\Omega_k}{2\pi} + (\theta^{2k-1}+\mathrm{i}\rho^{2k-1})(\theta^{2k}+\mathrm{i}\rho^{2k})\sinh\left(\frac{\mathrm{i}\Omega_k}{2\pi}\right)\right]\Big\}. \tag{4.43}$$

Thus

$$\operatorname{str}\exp[-\tilde{H}^2 t(0,0)] = \int_B \mathrm{d}^m\phi \prod_{k=1}^{m/2} \frac{\mathrm{i}\Omega_k}{2\pi}\frac{1}{\sinh(\mathrm{i}\Omega_k/2\pi)}\cosh\frac{\mathrm{i}\Omega_k}{4\pi}\operatorname{tr}\left[\exp\left(-\phi^i\phi^j\frac{F_{ij}}{2\pi}\right)\right]. \tag{4.44}$$

Hence, using (4.14), (4.32) and (4.36),

$$\operatorname{str}\exp[-H(p,p)]\,\mathrm{dvol} = \left[\operatorname{tr}\exp\left(\frac{-F}{2\pi}\right)\det\left(\frac{\mathrm{i}\Omega/2\pi}{\tanh\mathrm{i}\Omega/2\pi}\right)^{\frac{1}{2}}\right]\Bigg|_p \tag{4.45}$$

as required.

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